

Metoda de integrare prin părți

Formula de integrare prin părți :

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Exercițiu rezolvat:

1) $\int (5x-8)\sin x dx$

$$f(x) = 5x-8 \Rightarrow f'(x) = 5$$

$$g'(x) = \sin x \Rightarrow g(x) = \int \sin x dx = -\cos x$$

$$\begin{aligned}\text{Aplicăm formula de integrare : } \int (5x-8)\sin x dx &= (5x-8)(-\cos x) - \int 5(-\cos x) dx \\ &= -(5x-8)\cos x + 5 \int \cos x dx = -(5x-8)\cos x + 5\sin x + C\end{aligned}$$

Exerciții propuse:

2) $\int (3x+2)\cos x dx$; 3) $\int (-7x+3)e^x dx$; 4) $\int (-3x-1)\sin x dx$

Exercițiu rezolvat:

5) $\int (x^2-8x+6)e^x dx$

$$f(x) = x^2-8x+6 \Rightarrow f'(x) = 2x-8$$

$$g'(x) = e^x \Rightarrow g(x) = \int e^x dx = e^x$$

$$\text{Aplicăm formula de integrare : } \int (x^2-8x+6)e^x dx = (x^2-8x+6)e^x - \int (2x-8)e^x dx$$

$$\text{aplicăm încă o dată integrarea prin părți pentru } \int (2x-8)e^x dx$$

$$f(x) = 2x-8 \Rightarrow f'(x) = 2$$

$$g'(x) = e^x \Rightarrow g(x) = \int e^x dx = e^x$$

$$\int (x^2-8x+6)e^x dx = (x^2-8x+6)e^x - \left((2x-8)e^x - \int 2e^x dx \right)$$

$$= (x^2-8x+6)e^x - (2x-8)e^x + 2 \int e^x dx$$

Prin urmare :

$$= (x^2-8x+6)e^x - (2x-8)e^x + 2e^x + C = (x^2-8x+6-2x+8+2)e^x + C$$

$$= (x^2-10x+16)e^x + C$$

Exerciții propuse:

6) $\int (3x^2+2x-5)\cos x dx$; 7) $\int (-7x^2+3)e^x dx$; 8) $\int (x^2-3x+1)\sin x dx$

Exercițiu rezolvat:

9) $\int x^3 \ln x dx$

$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$

$$\Rightarrow \int x^3 \ln x dx = \frac{x^4}{4} \ln x - \int \frac{1}{x} \cdot \frac{x^4}{4} dx$$

$$g'(x) = x^3 \Rightarrow g(x) = \int x^3 dx = \frac{x^4}{4}$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 dx = \frac{x^4}{4} \ln x - \frac{1}{4} \cdot \frac{x^4}{4} + C$$

Exerciții propuse:

10) $\int x^2 \ln x dx$; 11) $\int x \ln x dx$; 12) $\int x^9 \ln x dx$; 13) $\int \ln x dx$; 14) $\int (x^3-6x)\ln x dx$

Exercițiu rezolvat:

$$15) I = \int \sqrt{3-x^2} dx = \int \frac{3-x^2}{\sqrt{3-x^2}} dx = \underbrace{\int \frac{3}{\sqrt{3-x^2}} dx}_{I_1} - \underbrace{\int \frac{x^2}{\sqrt{3-x^2}} dx}_{I_2}$$

$$I_1 = \int \frac{3}{\sqrt{3-x^2}} dx = 3 \int \frac{1}{\sqrt{3-x^2}} dx = 3 \arcsin \frac{x}{\sqrt{3}}$$

$$I_2 = \int \frac{x^2}{\sqrt{3-x^2}} dx = \int x \frac{x}{\sqrt{3-x^2}} dx$$

$$f(x) = x \Rightarrow f'(x) = 1$$

$$g'(x) = \frac{x}{\sqrt{3-x^2}} \Rightarrow g(x) = \int \frac{x}{\sqrt{3-x^2}} dx = -\sqrt{3-x^2}$$

$$I_2 = -x\sqrt{3-x^2} - \int 1(-\sqrt{3-x^2}) dx = -x\sqrt{3-x^2} + \int \sqrt{3-x^2} dx = -x\sqrt{3-x^2} + I$$

$$I = I_1 - I_2$$

$$I = 3 \arcsin \frac{x}{\sqrt{3}} - (-x\sqrt{3-x^2} + I)$$

Deci $I = 3 \arcsin \frac{x}{\sqrt{3}} + x\sqrt{3-x^2} - I$

$$2I = 3 \arcsin \frac{x}{\sqrt{3}} + x\sqrt{3-x^2} \Rightarrow I = \frac{3}{2} \arcsin \frac{x}{\sqrt{3}} + \frac{x}{2} \sqrt{3-x^2} + C$$

Exerciții propuse:

16) $\int \sqrt{2-x^2} dx$; 17) $\int \sqrt{x^2+2} dx$; 18) $\int \sqrt{x^2-7} dx$; 19) $\int \sqrt{x^2+5} dx$; 20) $\int \sqrt{x^2-1} dx$; 21) $\int \sqrt{1-x^2} dx$

Metoda de schimbare de variabilă

Formula de schimbare de variabilă :

$$\int f(u(x))u'(x) dx = F(u(x)) + C$$

unde F este o primitivă a lui f , și unde se poate nota: $u(x) = t$ și $u'(x) dx = dt$

$$\text{deci } \int f(t) dt = F(t) + C$$

Exercițiu rezolvat:

$$22) \int \frac{2x-1}{x^2-x+5} dx$$

$$u(x) = x^2 - x + 5 = t$$

$$u'(x) = 2x - 1$$

$$u'(x) dx = dt$$

$$(2x-1) dx = dt$$

$$\int \frac{dt}{t} = \int \frac{1}{t} dt = \ln|t| + C \Rightarrow \int \frac{2x-1}{x^2-x+5} dx = \ln|x^2-x+5| + C$$

Exerciții propuse:

23) $\int \frac{2x+1}{x^2+x+7} dx$; 24) $\int \frac{2x+4}{x^2+4x+5} dx$; 25) $\int \frac{4x-6}{x^2-3x+5} dx$; 26) $\int \frac{2x}{x^2+5} dx$; 27) $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Exercițiu rezolvat:

$$27) \int \frac{\sin x}{1 + \cos^2 x} dx = - \int \frac{-\sin x}{1 + \cos^2 x} dx$$

$$u(x) = \cos x = t$$

$$u'(x) = -\sin x$$

$$u'(x) dx = dt$$

$$-\sin x dx = dt$$

$$- \int \frac{dt}{1+t^2} = - \int \frac{1}{1+t^2} dt = -\arctg t + C$$

$$\Rightarrow - \int \frac{-\sin x}{1 + \cos^2 x} dx = -\arctg(\cos x) + C$$

Exerciții propuse:

$$28) \int \frac{\cos x}{1 + \sin^2 x} dx; 29) \int \frac{3x^2}{1 + (x^3)^2} dx; 30) \int \frac{2x}{1 + x^4} dx; 31) \int \frac{e^x}{1 + (e^x)^2} dx$$

Exercițiu rezolvat:

$$32) \int (2x-3)e^{x^2-3x+1} dx$$

$$u(x) = x^2 - 3x + 1 = t$$

$$u'(x) = 2x - 3$$

$$u'(x) dx = dt$$

$$(2x-3) dx = dt$$

$$\int e^t dt = e^t + C$$

$$\Rightarrow \int (2x-3)e^{x^2-3x+1} dx = e^{x^2-3x+1} + C$$

Exerciții propuse:

$$33) \int (2x+1)e^{x^2+x+4} dx; 34) \int (3x^2+2x+1)e^{x^3+x^2+x-5} dx; 35) \int (2x+1)2^{x^2+x+4} dx; 36) \int 2x \cdot 5^{x^2+4} dx$$

Exercițiu rezolvat:

$$37) \int \frac{\ln^3 x}{x} dx$$

$$u(x) = \ln x = t$$

$$u'(x) = \frac{1}{x}$$

$$u'(x) dx = dt$$

$$\frac{1}{x} dx = dt$$

$$\int t^3 dt = \frac{t^4}{4} + C \Rightarrow \int \frac{\ln^3 x}{x} dx = \frac{(\ln x)^4}{4} + C$$

Exerciții propuse:

$$38) \int \frac{\ln^5 x}{x} dx; 39) \int \frac{\ln x}{x} dx; 40) \int \frac{\ln^2 x}{x} dx$$

Exercițiu rezolvat:

$$41) \int (-3x-4)^5 dx = \frac{1}{-3} \int (-3)(-3x-4)^5 dx$$

$$u(x) = -3x - 4 = t$$

$$u'(x) = -3$$

$$u'(x) dx = dt$$

$$-3 dx = dt$$

$$\frac{1}{-3} \int t^5 dt = \frac{1}{-3} \cdot \frac{t^6}{6} + C \Rightarrow \frac{1}{-3} \int (-3)(-3x-4)^5 dx = \frac{1}{-3} \cdot \frac{(-3x-4)^6}{6} + C$$

Exerciții propuse:

42) $\int (4x-2)^3 dx$; 43) $\int (2x+5)^7 dx$; 44) $\int (-2x-3)^4 dx$; 45) $\int (-x+2)^9 dx$

Exercițiu rezolvat:

$$46) \int \frac{x^2}{\sqrt{x^6+1}} dx = \frac{1}{3} \int \frac{3x^2}{\sqrt{(x^3)^2+1}} dx$$

Observăm că $x^6 = (x^3)^2$ și că $(x^3)' = 3x^2$, deci

$$u(x) = x^3 = t \quad \frac{1}{3} \int \frac{dt}{\sqrt{t^2+1}} = \frac{1}{3} \ln(t + \sqrt{t^2+1}) + C \Rightarrow$$

$$u'(x) = 3x^2$$

$$u'(x) dx = dt \quad \frac{1}{3} \int \frac{3x^2}{\sqrt{(x^3)^2+1}} dx = \frac{1}{3} \ln(x^3 + \sqrt{(x^3)^2+1}) + C = \frac{1}{3} \ln(x^3 + \sqrt{x^6+1}) + C$$

$$3x^2 dx = dt$$

Exerciții propuse:

47) $\int \frac{x^2}{\sqrt{x^6+7}} dx$; 48) $\int \frac{x^2}{\sqrt{x^6-1}} dx$; 49) $\int \frac{x^2}{\sqrt{x^6-5}} dx$; 50) $\int \frac{x}{\sqrt{x^4+1}} dx$; 51) $\int \frac{x}{\sqrt{x^4-3}} dx$

Exercițiu rezolvat:

$$52) \int \frac{x}{\sqrt{2-x^4}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{2-(x^2)^2}} dx$$

$$u(x) = x^2 = t \quad \frac{1}{2} \int \frac{dt}{\sqrt{2-t^2}} = \frac{1}{2} \arcsin \frac{t}{\sqrt{2}} + C \Rightarrow$$

$$u'(x) = 2x$$

$$u'(x) dx = dt \quad \frac{1}{2} \int \frac{2x}{\sqrt{1-(x^2)^2}} dx = \frac{1}{2} \arcsin \frac{x^2}{\sqrt{2}} + C$$

$$2x dx = dt$$

Exerciții propuse:

53) $\int \frac{x}{\sqrt{4-x^4}} dx$; 54) $\int \frac{x}{\sqrt{1-x^4}} dx$; 55) $\int \frac{x}{\sqrt{5-x^4}} dx$; 56) $\int \frac{x^2}{\sqrt{4-x^6}} dx$; 57) $\int \frac{x^2}{\sqrt{2-x^6}} dx$

Exerciții rezolvate:

$$58) \int \frac{1}{x-3} dx$$

$$u(x) = x-3 = t$$

$$u'(x) = 1$$

$$u'(x) dx = dt$$

$$1 dx = dt$$

$$\int \frac{1}{t} dt = \ln|t| + C \Rightarrow$$

$$\int \frac{1}{x-3} dx = \ln|x-3| + C$$

$$59) \int \frac{1}{(x+4)^3} dx$$

$$u(x) = x+4 = t$$

$$u'(x) = 1$$

$$u'(x) dx = dt$$

$$1 dx = dt$$

$$\int \frac{1}{t^4} dt = \int t^{-4} dt = \frac{t^{-3}}{-3} + C \Rightarrow$$

$$\int \frac{1}{(x+4)^3} dx = \frac{(x+4)^{-3}}{-3} + C$$

Exerciții propuse:

60) $\int \frac{1}{x-1} dx$; 61) $\int \frac{1}{x-6} dx$; 62) $\int \frac{1}{x+3} dx$; 63) $\int \frac{1}{x+1} dx$

64) $\int \frac{1}{(x+1)^3} dx$; 65) $\int \frac{1}{(x-4)^3} dx$; 66) $\int \frac{1}{(x+2)^5} dx$; 67) $\int \frac{1}{(x-5)^{12}} dx$