FIŞĂ _ REZOLVAREA COMPLETĂ A DIFERITELOR TIPURI DE EXERCIŢII
Clasa a XII-a _**Analiză matematică** Prof. Mestecan Cornelia
Matematică_bacalaureat (M_tehnologic, M_mate-info, M_st nat)
Exerciţiul 2/sub. III/varianta 9/ M_mate-info/2025

2. Se consideră funcţia $f:\mathbb{R}\to\mathbb{R}, f(x)=3x^2-1+e^{2x}$.

a) Arătaţi că $\int_0^3 \left(f(x)-e^{2x}\right)dx = 24$.

b) Arătaţi că $\int_0^1 4x\left(f(x)-3x^2+1\right)dx = e^2+1$.

c) Demonstraţi că $\lim_{x\to 0}\frac{1}{x^2}\int_0^x \frac{f(t)}{t+1}dt = 1$.

Rezolvare:

a)
$$\int_0^3 \left(f(x)-e^{2x}\right)dx = \int_0^3 \left(3x^2-1\cancel{+e^{2x}}\cancel{-e^{2x}}\right)dx = \int_0^3 \left(3x^2-1\right)dx$$
$$=\left(\cancel{3}\frac{x^3}{\cancel{3}}-x\right)\Bigg|_0^3 = \left(x^3-x\right)\Big|_0^3 = (27-3)-(0-0)=24, (A)$$

Am folosit formulele: $\int x^n dx = \frac{x^{n+1}}{n+1}+C$, $\int 1dx = x+C$ şi $\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a)$

b)
$$\int_0^1 4x\left(f(x)-3x^2+1\right)dx = \int_0^1 4x\left(\cancel{3x^2}\cancel{-1}+e^{2x}\cancel{-3x^2}\cancel{+1}\right)dx$$
$$=\int_0^1 4xe^{2x}dx = 4\int_0^1 xe^{2x}dx$$
$$\int_0^1 xe^{2x}dx = ?$$

Metoda 1.

$$f(x)=x \to f'(x)=1$$
$$g'(x)=e^{2x} \to g(x)=\frac{1}{2}e^{2x}$$

$$\int_a^b f(x)\cdot g'(x)dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f'(x)\cdot g(x)dx$$
$$\int_0^1 xe^{2x}dx = \frac{1}{2}xe^{2x}\Big|_0^1 - \frac{1}{2}\int_0^1 e^{2x}dx$$
$$\overset{2)}{=} \frac{1}{2}\left(e^2-0\right)-\frac{1}{4}e^{2x}\Big|_0^1 = \frac{2e^2}{4}-\frac{e^2}{4}+\frac{1}{4}=\frac{1}{4}\left(e^2+1\right)$$

$$\Rightarrow 4\int_0^1 xe^{2x}dx = e^2+1,(A)$$

Metoda 2.

$$f(x)=x \to f'(x)=1 \qquad \int f(x)\cdot g'(x)dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)dx$$

$$g'(x)=e^{2x} \to g(x)=\frac{1}{2}e^{2x} \qquad \int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}+C$$

$$\int_0^1 xe^{2x}dx = \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)\Bigg|_0^1 = \left(\frac{e^2}{2}-\frac{e^2}{4}\right) - \left(0-\frac{1}{4}\right) = \frac{e^2}{4}+\frac{1}{4} = \frac{e^2+1}{4}$$

$$\Rightarrow 4\int_0^1 xe^{2x}dx = e^2+1,(A)$$

c) Folosim teorema de existenţă a primitivelor unei funcţii continue

(Dacă $f:[a,b]\to\mathbb{R}$ este o funcţie continuă, atunci funcţia $F:[a,b]\to\mathbb{R}$, $F(x)=\int_a^x f(t)dt$

este primitiva lui *f* care se anulează în *a*, adică $\left(\int_a^x f(t)dt\right)' = f(x), \forall x\in[a,b]$.)

Prin urmare:

$$\lim_{x\to 0}\frac{1}{x^2}\int_0^x \frac{f(t)}{t+1}dt \underset{l'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0}\frac{\left(\int_0^x \frac{f(t)}{t+1}dt\right)'}{\left(x^2\right)'} = \lim_{x\to 0}\frac{f(x)}{2x} \underset{l'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0}\frac{f'(x)}{2}$$

$$=\lim_{x\to 0}\frac{1}{2}\left(6x+2e^{2x}\right) = \frac{1}{2}\cdot 2 = 1,(A)$$

Exerciţiul 2/sub. III/varianta 9/ M_ șt-nat/2025

2. Se consideră funcţia $f:(0,+\infty)\to\mathbb{R}$, $f(x)=\frac{x-1}{\sqrt{x}}$.

a) Arătaţi că $\int_2^4 f(x)\sqrt{x}dx = 4$.

b) Arătaţi că $\int_1^4 f(x)dx = \frac{8}{3}$.

c) Arătaţi că volumul corpului obţinut prin rotaţia graficului funcţiei $g:[2,3]\to\mathbb{R}$,

$g(x)=\dfrac{\sqrt{2}}{f(x)}$, în jurul axei Ox este egal cu $\pi\ln(4e)$.

Rezolvare:

a) $$\int_2^4 f(x)\sqrt{x}dx=\int_2^4 \frac{x-1}{\cancel{\sqrt{x}}_{1}}\cancel{\sqrt{x}}\,dx=\int_2^4(x-1)\,dx=\left(\frac{x^2}{2}-x\right)\Bigg|_2^4=\left(\frac{16}{2}-4\right)-\left(\frac{4}{2}-2\right)=4-0=4,(A)$$

Am folosit formulele: $\int x^n dx=\dfrac{x^{n+1}}{n+1}+C$, $\int 1dx=x+C$ şi $\int_a^b f(x)dx=F(x)\big|_a^b=F(b)-F(a)$

$$\int_1^4 f(x)dx=\int_1^4\frac{x-1}{\sqrt{x}}dx=\int_1^4\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx=\int_1^4\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)dx$$

b) $$=\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)\Bigg|_1^4=\left(\frac{2}{3}x\sqrt{x}-2\sqrt{x}\right)\Bigg|_1^4=\left(\frac{16}{3}-{}^{3)}4\right)-\left(\frac{2}{3}-{}^{3)}2\right)=\frac{4}{3}+\frac{4}{3}=\frac{8}{3},(A)$$

$$V=\pi\int_2^3 g^2(x)\,dx$$

c) $$V=\pi\int_2^3\left(\frac{\sqrt{2}}{\frac{x-1}{\sqrt{x}}}\right)^2dx=\pi\int_2^3\left(\frac{\sqrt{2}\cdot\sqrt{x}}{x-1}\right)^2dx\Rightarrow V=\pi\int_2^3\frac{2x}{(x-1)^2}dx$$

$$\Rightarrow V=2\pi\int_2^3\frac{x-1+1}{(x-1)^2}dx=2\pi\int_2^3\frac{1}{x-1}dx+2\pi\int_2^3(x-1)^{-2}dx$$

$$V=2\pi\left(\ln(x-1)\right)\Big|_2^3-2\pi\frac{1}{x-1}\Bigg|_2^3=2\pi(\ln 2-\ln 1)-\left(2\pi\cdot\frac{1}{2}-2\pi\right)$$

$$V=2\pi\ln 2+\pi=\pi\ln 4+\pi\ln e=\pi\ln 4e,(A)$$

Am folosit formulele: $\int x^n dx=\dfrac{x^{n+1}}{n+1}+C$, $\int\dfrac{1}{x-a}dx=\ln|x-a|+C$ şi

$\int_a^b f(x)dx=F(x)\big|_a^b=F(b)-F(a)$

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Exerciţiul 2/sub. III/varianta 9/ M_ tehnologic/2025

2. Se consideră funcţia $f:(0,+\infty)\to\mathbb{R}$, $f(x)=x-1+2\ln x$.

a) Arătaţi că $\int_1^3 (f(x)-2\ln x)dx=2$.

b) Arătaţi că $\int_1^e \left(\frac{f(x)-x+1}{x}\right)dx=1$.

c) Determinaţi numărul real *a* , ştiind că aria suprafeţei plane delimitate de graficul funcţiei $g:(0,+\infty)\to\mathbb{R}$, $g(x)=3x(f(x)+1)$, axa *Ox* şi dreptele de ecuaţii $x=1$ şi $x=3$ este egală cu $a+27\ln 3$.

Rezolvare:

a)
$$\int_1^3 (f(x)-2\ln x)dx=\int_1^3 (x-1+2\ln x-2\ln x)dx=\int_1^3 (x-1)dx=\left(\frac{x^2}{2}-x\right)\Bigg|_1^3$$
$$=\left(\frac{9}{2}-3\right)-\left(\frac{1}{2}-1\right)=\frac{3}{2}+\frac{1}{2}=\frac{4}{2}=2,(A)$$

Am folosit formulele: $\int x^n dx=\frac{x^{n+1}}{n+1}+C$, $\int 1dx=x+C$ şi $\int_a^b f(x)dx=F(x)\Big|_a^b=F(b)-F(a)$

b) $$\int_1^e \left(\frac{f(x)-x+1}{x}\right)dx=\int_1^e \left(\frac{\cancel{x}\cancel{-1}+2\ln x\cancel{-x}\cancel{+1}}{x}\right)dx=2\int_1^e \frac{1}{x}\cdot(\ln x)dx$$

$$\left.\begin{array}{l} u(x)=\ln x=t \\ u'(x)=\frac{1}{x} \\ u'(x)dx=dt \end{array}\right\}\Rightarrow \int t dt=\frac{t^2}{2}+C\Rightarrow \int \frac{1}{x}\cdot(\ln x)dx=\frac{(\ln x)^2}{2}+C$$

$$\Rightarrow 2\int_1^e \frac{1}{x}\cdot(\ln x)dx=2\left(\frac{(\ln x)^2}{2}\right)\Bigg|_1^e=2\left(\frac{(\ln e)^2}{2}-\frac{(\ln 1)^2}{2}\right)$$
$$=2\left(\frac{1^2}{2}-\frac{0^2}{2}\right)=1,(A)$$

c) aria suprafeţei plane delimitate de graficul funcţiei $g : I \to \mathbb{R}$, axa *Ox* şi dreptele de ecuaţii $x = a$ şi $x = b$ este $A = \int_a^b g(x)\,dx$.

În cazul nostru avem $g : (0, +\infty) \to \mathbb{R}$, $g(x) = 3x\big(f(x) + 1\big)$, axa *Ox* şi dreptele de ecuaţii $x = 1$ şi $x = 3$

$$A = \int_1^3 g(x)\,dx = \int_1^3 3x\big(f(x)+1\big)dx = \int_1^3 3x\big(x \cancel{-1} + 2\ln x \cancel{+1}\big)dx$$

$$= \int_1^3 3x(x + 2\ln x)\,dx = \underbrace{\int_1^3 3x^2 dx}_{I_1} + \underbrace{\int_1^3 6x(\ln x)\,dx}_{I_2}$$

$$I_1 = \int_1^3 3x^2 dx = 3\int_1^3 x^2 dx = \cancel{3}\,\frac{x^3}{\cancel{3}}\Bigg|_1^3 = 27 - 1 = 26$$

$$I_2 = \int_1^3 6x \ln x\,dx = 6\int_1^3 x\ln x\,dx$$

Folosim metoda de integrare prin părţi:

Metoda 1. $\int x \ln x\,dx = ?$

$$f(x) = \ln x \to f'(x) = \frac{1}{x}$$

$$g'(x) = x \to g(x) = \frac{x^2}{2}$$

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx$$

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}}{2}dx = \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + C = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$$\Rightarrow I_2 = 6\int_1^3 x\ln x\,dx = 6\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Bigg|_1^3 = 6\left(\frac{9}{2}\ln 3 - \frac{9}{4}\right) - 6\left(\frac{1}{2}\underbrace{\ln 1}_{0} - \frac{1}{4}\right)$$

$$I_2 = 27\ln 3 - \frac{27}{2} + \frac{3}{2} = 27\ln 3 - 12$$

$$\left.\begin{array}{l} A = I_1 + I_2 \\ a + 27\ln 3 = 26 + 27\ln 3 - 12 \\ a + 27\ln 3 = 14 + 27\ln 3 \end{array}\right\} \Rightarrow a = 14$$

Metoda 2. $\int_1^3 x\ln x dx = ?$

$$\int_a^b f(x)\cdot g'(x)dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b f'(x)\cdot g(x)dx$$

$$f(x) = \ln x \to f'(x) = \frac{1}{x}$$

$$g'(x) = x \to g(x) = \frac{x^2}{2}$$

$$\int_1^3 x\ln x dx = \frac{x^2}{2}\ln x\Big|_1^3 - \int_1^3 \frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}\ln x\Big|_1^3 - \frac{1}{2}\int_1^3 xdx$$

$$= \left(\frac{9}{2}\ln 3 - \frac{1}{2}\underbrace{\ln 1}_{0}\right) - \frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^3 = \frac{9}{2}\ln 3 - \left(\frac{9}{4} - \frac{1}{4}\right) = \frac{9}{2}\ln 3 - 2$$

$$\Rightarrow I_2 = 6\int_1^3 x\ln x dx = 6\cdot\frac{9}{2}\ln 3 - 12 = 27\ln 3 - 12$$

Exerciţiul 2/sub. III/varianta 5/ M_mate-info/2025

2. Se consideră funcţia $f:(0,+\infty)\to\mathbb{R}$, $f(x) = 8x^3 + 1 + \ln^2 x$.

a) Arătaţi că $\int_1^2 \left(f(x) - \ln^2 x\right)dx = 31$.

b) Arătaţi că $\int_1^e \frac{f(x) - 8x^3 - 1}{x}dx = \frac{1}{3}$.

c) Pentru fiecare număr natural nenul n se consideră numărul natural $I_n = \int_1^{e^2} \frac{x(\ln x)^n}{f(x) - 8x^3}dx$.

Demonstraţi că $I_{n+2} + I_n \le 2^{n-1}\left(e^4 - 1\right)$, pentru orice număr natural nenul n.

Rezolvare:

a)

$$\int_1^2 \left(f(x) + \ln^2 x\right)dx = \int_1^2 \left(8x^3 + 1 - \ln^2 x + \ln^2 x\right)dx = \int_1^2 \left(8x^3 + 1\right)dx$$

$$= \left(\overset{2}{\cancel{8}}\frac{x^4}{\cancel{4}} + x\right)\Big|_1^2 = (32 + 2) - (2 + 1) = 31, (A)$$

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Am folosit formulele: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $\int 1dx = x + C$ şi $\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$

b) $\int_1^e \frac{f(x) - 8x^3 - 1}{x} dx = \int_1^e \frac{\cancel{8x^3} \cancel{+1} + \ln^2 x \cancel{-8x^3} \cancel{-1}}{x} dx = \int_1^e \frac{\ln^2 x}{x} dx$

$$\left.\begin{aligned} &u(x) = \ln x = t \\ &u'(x) = \frac{1}{x} \\ &u'(x)dx = dt \end{aligned}\right\} \Rightarrow \int t^2 dt = \frac{t^3}{3} + C \Rightarrow \int \frac{1}{x} \cdot (\ln x)^2 dx = \frac{(\ln x)^3}{3} + C$$

$$\Rightarrow \int_1^e \frac{\ln^2 x}{x} dx = \frac{(\ln x)^3}{3}\Bigg|_1^e = \frac{(\ln e)^3}{3} - \frac{(\ln 1)^3}{3} = \frac{1}{3}, (A)$$

c) $I_n = \int_1^{e^2} \frac{x(\ln x)^n}{f(x) - 8x^3} dx$ să se demonstreze că $I_{n+2} + I_n \le 2^{n-1}(e^4 - 1)$, $\forall n \in \mathbb{N}^*$

$$I_n = \int_1^{e^2} \frac{x(\ln x)^n}{\cancel{8x^3} + 1 + \ln^2 x \cancel{-8x^3}} dx = \int_1^{e^2} \frac{x(\ln x)^n}{1 + \ln^2 x} dx$$

$$I_{n+2} + I_n = \int_1^{e^2} \frac{x(\ln x)^{n+2}}{1 + \ln^2 x} dx + \int_1^{e^2} \frac{x(\ln x)^n}{1 + \ln^2 x} dx = \int_1^{e^2} \frac{x(\ln x)^n \cancel{(\ln^2 x + 1)}}{\cancel{1 + \ln^2 x}} dx = \int_1^e x(\ln x)^n dx$$

$$x \in [1, e^2] \Leftrightarrow 1 \le x \le e^2 \Rightarrow \ln 1 \le \ln x \le \ln e^2 \Leftrightarrow 0 \le \ln x \le 2$$

$$\Rightarrow I_{n+2} + I_n \le \int_1^{e^2} x \cdot 2^n dx$$

$$I_{n+2} + I_n \le 2^n \cdot \frac{x^2}{2}\Bigg|_1^{e^2} \Rightarrow I_{n+2} + I_n \le 2^{n-1}(e^4 - 1), (A), \forall n \in \mathbb{N}^*$$