FIŞĂ _RECAPITULARE NOŢIUNI TEORETICE ŞI EXEMPLE DE EXERCIŢII

Clasa a XII-a _**Analiză matematică** Prof. Mestecan Cornelia

Matematică (M_tehnologic, M_mate-info, M_st nat)

INTEGRAREA FUNCŢIILOR RAŢIONALE

Recapitulare noţiuni teoretice

Funcţii raţionale simple: $\frac{A}{(x-a)^n}, n \in \mathbb{N}^*$; $\frac{Ax+B}{(ax^2+bx+c)^n}, n \in \mathbb{N}^*, \Delta = b^2-4ac<0$

Integrarea funcţiilor raţionale simple:

1. $\int \frac{1}{x-a}\,dx = \ln|x-a| + C$

Exemple: $\int \frac{1}{x+3}\,dx = \ln|x+3| + C$; $\int \frac{5}{x-3}dx = 5\int \frac{1}{x-3}\,dx = 5\ln|x-3| + C$

Exerciţii propuse:

1) $\int \frac{1}{x+1}\,dx$; 2) $\int \frac{1}{x-8}\,dx$; 3) $\int \frac{2}{x-1}\,dx$; 4) $\int \frac{9}{x-11}\,dx$; 5) $\int \frac{3}{x+2}\,dx$

2. $\int \frac{1}{(x-a)^n}\,dx = \int (x-a)^{-n}\,dx = \frac{(x-a)^{-n+1}}{-n+1} + C = -\frac{1}{n-1}\cdot\frac{1}{(x-a)^{n-1}}, n \geq 2$

Exemple: $\int \frac{1}{(x-1)^2}\,dx = \int (x-1)^{-2}\,dx = \frac{(x-1)^{-2+1}}{-2+1} + C = -\frac{1}{x-1} + C$

$$\int \frac{7}{(x+3)^5}\,dx = 7\int (x+3)^{-5}\,dx = 7\frac{(x+3)^{-5+1}}{-5+1} + C = -\frac{7}{4}\cdot\frac{1}{(x+3)^4} + C$$

Exerciţii propuse:

6) $\int \frac{1}{(x+1)^3}\,dx$; 7) $\int \frac{1}{(x-8)^2}\,dx$; 8) $\int \frac{2}{(x-1)^7}\,dx$; 9) $\int \frac{9}{(x-11)^3}\,dx$; 10) $\int \frac{3}{(x+2)^9}\,dx$

3.

$$\int \frac{1}{ax^2+bx+c}\,dx = \int \frac{1}{a\left(x+\frac{b}{2a}\right)^2 + \frac{-\Delta}{4a}}dx = \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2 + \frac{-\Delta}{4a^2}}dx = \frac{1}{a}\,\frac{1}{\frac{\sqrt{-\Delta}}{2a}}\,arctg\,\frac{x+\frac{b}{2a}}{\frac{\sqrt{-\Delta}}{2a}} + C$$

$$= \frac{1}{a}\cdot\frac{2a}{\sqrt{-\Delta}}\,arctg\,\frac{2ax+b}{\sqrt{-\Delta}} + C = \frac{2}{\sqrt{-\Delta}}\,arctg\,\frac{2ax+b}{\sqrt{-\Delta}} + C;\ldots unde..\Delta<0$$

s-a folosit faptul că dacă notăm $u(x)=\left(x+\frac{b}{2a}\right)=t \Rightarrow u'(x)=1$ și atunci $u'(x)dx=dt \Leftrightarrow 1dx=dt$

$$\frac{1}{a}\int\frac{1}{\left(x+\frac{b}{2a}\right)^2+\frac{-\Delta}{4a^2}}dx \Rightarrow \frac{1}{a}\int\frac{1}{t^2+\frac{-\Delta}{4a^2}}dt=\frac{1}{a}\cdot\frac{1}{\sqrt{\frac{-\Delta}{4a^2}}}arctg\frac{t}{\sqrt{\frac{-\Delta}{4a^2}}}+C$$

$$=\frac{1}{a}\cdot\frac{1}{\frac{\sqrt{-\Delta}}{2a}}arctg\frac{t}{\frac{\sqrt{-\Delta}}{2a}}+C$$

Exemple:

$$\int\frac{5}{x^2+x+3}dx=5\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{11}{4}}dx=5\cdot\frac{1}{\sqrt{\frac{11}{4}}}arctg\frac{x+\frac{1}{2}}{\sqrt{\frac{11}{4}}}+C=\frac{10}{\sqrt{11}}arctg\frac{2x+1}{\sqrt{11}}+C$$

$$\Delta=b^2-4ac=1-12=-11$$

$$-\Delta=11$$

$$\int\frac{7}{3x^2-2x+2}dx=7\int\frac{1}{3\left(x-\frac{2}{6}\right)^2+\frac{20}{4}}dx=\frac{7}{3}\int\frac{1}{\left(x-\frac{1}{3}\right)^2+5}=\frac{7}{3}\cdot\frac{1}{\sqrt{5}}arctg\frac{x-\frac{1}{3}}{\sqrt{5}}+C$$

$$=\frac{7}{3\sqrt{5}}arctg\frac{3x-1}{3\sqrt{5}}+C$$

$$\Delta=b^2-4ac=4-24=-20$$

$$-\Delta=20$$

Exerciţii propuse:

11) $\int\frac{1}{x^2+x+1}dx$; 12) $\int\frac{1}{x^2-x+2}dx$; 13) $\int\frac{3}{2x^2-3x+2}dx$; 14) $\int\frac{4}{5x^2+2x+3}dx$

4.

$$\int\frac{1}{(x-a)(x-b)}dx=\int\left(\frac{A}{x-a}+\frac{B}{x-b}\right)dx=A\int\frac{1}{x-a}dx+B\int\frac{1}{x-b}dx=A\ln|x-a|+B\ln|x-b|+C$$

s-a folosit descompunerea $\frac{1}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$ şi pentru a afla A şi B

folosim **metoda coeficienţilor nedeterminaţi (**aducem la acelaşi numitor şi folosim faptul că **dacă două fracţii egale au numitorii egali atunci au şi numărătorii egali):**

$$\frac{1}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}\Leftrightarrow 1=A(x-b)+B(x-a)\Leftrightarrow 0x+1=x(A+B)+(-bA-aB)$$

$$\Rightarrow\begin{cases}A+B=0\\-bA-aB=1\end{cases}\Leftrightarrow\begin{cases}A=-B\\bB-aB=1\end{cases}\Leftrightarrow\begin{cases}A=-B\\B=\frac{1}{b-a}\end{cases}\Leftrightarrow\begin{cases}A=-\frac{1}{b-a}\\B=\frac{1}{b-a}\end{cases}$$

Exemplu:

$$\int\frac{2}{(x-1)(x+3)}dx=2\int\left(\frac{\frac{1}{4}}{x-1}+\frac{-\frac{1}{4}}{x+3}\right)dx=\frac{2}{4}\int\frac{1}{x-1}dx-\frac{2}{4}\int\frac{1}{x+3}dx=\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+3|+C$$

$$\frac{1}{(x-a)(x-b)}=\frac{A}{x-1}+\frac{B}{x+3}\Leftrightarrow 1=A(x+3)+B(x-1)\Leftrightarrow 0x+1=x(A+B)+(3A-B)$$

$$\Rightarrow\begin{cases}A+B=0\\3A-B=1\end{cases}\Leftrightarrow\begin{cases}A=-B\\-3B-B=1\end{cases}\Leftrightarrow\begin{cases}A=-B\\B=\frac{1}{-4}\end{cases}\Leftrightarrow\begin{cases}A=\frac{1}{4}\\B=-\frac{1}{4}\end{cases}$$

Exerciţii propuse:

15) $\int\frac{1}{(x-2)(x-3)}dx$; 16) $\int\frac{1}{(x+2)(x-5)}dx$; 17) $\int\frac{3}{(x-4)(x+1)}dx$; 18) $\int\frac{6}{(x+3)(x+7)}dx$

5.

$$\int\frac{1}{ax^2+bx+c}dx=\int\frac{1}{a(x-x_1)(x-x_2)}dx=\frac{1}{a}\int\left(\frac{A}{x-x_1}+\frac{B}{x-x_2}\right)dx=\frac{1}{a}\int\frac{A}{x-x_1}dx+\frac{1}{a}\int\frac{B}{x-x_2}dx$$

$$=\frac{A}{a}\ln|x-x_1|+\frac{B}{a}\ln|x-x_2|+C$$

$$\Delta=b^2-4ac\geq 0\Rightarrow x_1=\frac{-b-\sqrt{\Delta}}{2a};x_2=\frac{-b+\sqrt{\Delta}}{2a}\Rightarrow ax^2+bx+c=a(x-x_1)(x-x_2)$$

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Exemplu:

$$\int \frac{2}{6x^2-5x+1}dx = 2\int \frac{1}{6\left(x-\frac{1}{3}\right)\left(x-\frac{1}{2}\right)}dx = \frac{2}{6}\int \frac{1}{\left(x-\frac{1}{3}\right)\left(x-\frac{1}{2}\right)}dx = \frac{1}{3}\int\left(\frac{A}{x-\frac{1}{3}}+\frac{B}{x-\frac{1}{2}}\right)dx$$

$$6x^2-5x+1=0$$

$$\Delta = 25-24=1 \Rightarrow x_1 = \frac{5-1}{12}=\frac{4}{12}=\frac{1}{3}; x_2=\frac{5+1}{12}=\frac{6}{12}=\frac{1}{2} \Rightarrow 6x^2-5x+1=6\left(x-\frac{1}{3}\right)\left(x-\frac{1}{2}\right)$$

s-a folosit descompunerea $\frac{1}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$ şi pentru a afla A şi B folosim **metoda coeficienţilor nedeterminaţi:**

$$\frac{1}{\left(x-\frac{1}{3}\right)\left(x-\frac{1}{2}\right)}=\frac{A}{x-\frac{1}{3}}+\frac{B}{x-\frac{1}{2}}$$ aducem la acelaşi numitor şi folosim faptul că **dacă două fracţii egale au numitorii egali atunci au şi numărătorii egali**

$$1=A\left(x-\frac{1}{2}\right)+B\left(x-\frac{1}{3}\right) \Leftrightarrow 1 = Ax-\frac{A}{2}+Bx-\frac{B}{3} \Leftrightarrow 0x+1=x(A+B)+\left(-\frac{A}{2}-\frac{B}{3}\right)$$

Deci

$$\Rightarrow \begin{cases} A+B=0 \\ -\frac{A}{2}-\frac{B}{3}=1 \end{cases} \Leftrightarrow \begin{cases} A=-B \\ \frac{^{3)}B}{2}-\frac{^{2)}B}{3}=1 \end{cases} \Leftrightarrow \begin{cases} A=-B \\ \frac{B}{6}=1 \end{cases} \Leftrightarrow \begin{cases} A=-B \\ B=6 \end{cases} \Leftrightarrow \begin{cases} A=-6 \\ B=6 \end{cases}$$

Prin urmare :

$$\frac{1}{3}\int\left(\frac{A}{x-\frac{1}{3}}+\frac{B}{x-\frac{1}{2}}\right)dx = \frac{1}{3}\int\left(\frac{-6}{x-\frac{1}{3}}+\frac{6}{x-\frac{1}{2}}\right)dx = \frac{1}{3}\int\frac{-6}{x-\frac{1}{3}}dx+\frac{1}{3}\int\frac{6}{x-\frac{1}{2}}dx = -2\int\frac{1}{x-\frac{1}{3}}dx+2\int\frac{1}{x-\frac{1}{2}}dx$$

$$=-2\ln\left|x-\frac{1}{3}\right|+2\ln\left|x-\frac{1}{2}\right|+C$$

Exerciţii propuse:

19) $\int\frac{1}{x^2+3x+2}dx$; 20) $\int\frac{1}{x^2-4x+3}dx$; 21) $\int\frac{3}{2x^2-3x-2}dx$; 22) $\int\frac{4}{5x^2+6x+1}dx$